Optimisation to Arithmetic Mean Sequence Intermediate Values Solver

Research

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ABSTRACT

Arithmetic Mean sequence is a list of numbers that given the first 2 numbers, the remaining numbers in the sequence can generated by taking the arithmetic mean of the previous 2 values. A problem arises when the sequence is known to be an arithmetic mean sequence, but only the first and last numbers in the list are known. Using simultaneous equations and variable replacement techniques the missing sequence values can be found. If the sequence is small, the lengthy process of simultaneous equations can be followed. Nonetheless, for decently sized sequences the equations can become cumbersome and time consuming to work with. The question than is, how can the solution be solved in a timely fashion without adding complexity. This paper presents 2 set of formulæ that improve provide an answer to the question imposed.

KEYWORDS

Average Sequence, Arithmetic Mean Sequence, Optimisation Formulæ, Arithmetic Mean Special Case

1 INTRODUCTION

On March 17, 2019, Billiant.org issued the following problem:

> For the sequence $45, a, b, c, d, 100$, where starting from b, the value is the arithmetic mean of the previous 2 values. Find the value of a.

The above question can be solved by using the Arithmetic Mean equation for 2 numbers

$$\text{Arithemtic Mean} = \frac{a+b}{2}, \text{for } a \text{ and } b \in \mathbb{R} \tag{1}$$

Using Equation 1, the question imposed by Brilliant.org can be solved by solving for a the equation

$$100 = \frac{\frac{a+\frac{45+a}{2}}{2} + \frac{\frac{45+a}{2}+\frac{a+\frac{45+a}{2}}{2}}{2}}{2} \tag{2}$$

Within some time, Equation 2 can be solved for a to arrive to the value 125.

Solving the question is interesting practice. Taking to another level the other variables b, c and d can also be found. Using equation 1

$$\begin{aligned} b &= \frac{45+a}{2} & c &= \frac{a+b}{2} & d &= \frac{b+c}{2} \\ &= \frac{45+125}{2} & &= \frac{125+85}{2} & &= \frac{85+105}{2} \\ &= 85 & &= 105 & &= 95 \end{aligned}$$

Interesting exercise. Analysing the steps followed for determining the results, 2 questions arise.

(1) What if the number of intermediate variables in the sequence is big? Finding the value of a will require exponential time to solve.
(2) What if another variable, not a is required? Can the value be determined without the need to find all previous values?

The next sections attempt to solve the 2 questions imposed. While proofs for the conclusions in sections §2 and §3 are provided in section §4.

2 OPTIMISING RECURSIVE ARITHMETIC MEAN FORMULA

Equation 2 can become complex and time consuming, $O(n^2)$, as the number of intermediate values increases in the sequence. This can be visualised by outlining the algorithm that one will follow in solving the equation. Let x and y be the start and end values in the sequence and var_n be the n^{th} number in the sequence of unknown values.

Algorithm 1: High level steps for solving for var_1

Input: $x, var_1, var_2, var_3, \ldots, y$
Output: Value for var_1

1. Staring from $y = \frac{var_{n-1}+var_n}{2}$
2. Expand var_{n-1} and var_n until they are expressed as fractions within fractions in terms of value x and var_1
3. Eliminate the fractions by multiplying with 2 until the equation ends up in the form
$$Ay = Bvar_1 + Cx$$
where A, B and C are some constant
4. Finally by making var_1 the subject of the equation, the required value can be found
$$var_1 = \frac{Ay - Cx}{B} \tag{3}$$

Note the repeating nature of steps 2 and 3 in Algorithm 1. The repetitions will become more and more intensive as the number of intermediate unknown values increases.

2.1 Simplifying for var_1

The original question imposed was about finding the first unknown value in the sequence, which is not an arithmetic mean of the previous 2 values.

Let's assume that x and y are the first and last number in the sequence and are known values. Then if $|var|$ is the number of intermediate unknown variables in the sequence, var_1 can be found using algorithm 1 or if a repertoire like the one below is provided, the required formula can be used.

$$\begin{aligned} |var| &= 1, & var_1 &= 2y - x \\ |var| &= 2, & var_1 &= \frac{4y-x}{3} \\ |var| &= 3, & var_1 &= \frac{8y-3x}{5} \\ |var| &= 4, & var_1 &= \frac{16y-5x}{11} \\ |var| &= 5, & var_1 &= \frac{32y-11x}{21} \end{aligned}$$

Having such a repertoire at one's disposal is interesting but what if the formula for the number of variables is not in the list? One can look at generalising the formula. Starting by looking at the variable multipliers and the denominator of each formula the below table can be constructed to determine a pattern.

Table 1: Multipliers and Divisors for the equation of a

$\|vars\|$	1	2	3	4	5
x	1	1	3	5	11
y	2	4	8	16	32
denominator	1	3	5	11	21

From table 1 it can be concluded that:

(1) The multiplier of x is the Jacobsthal Number [1, 2] for n
(2) The multiplier of y is 2^n
(3) The denominator is the Jacobsthal Number for $n + 1$

Using equation 3 from the repository and the conclusions from the table, it can be formulated that var_1 can be determined using the equation:

$$var_1 = \frac{2^{|vars|}y - J_{|vars|}x}{J_{|vars|+1}} \tag{4}$$

2.2 Finding var_n for $n \geq 2$ in terms of x and var_1

Given that the first 2 values of the sequence are known, it is possible to determine the remaining variables. Simplifying Equation 2 for each unknown variable:

$$\begin{aligned} var_2 &= \frac{x + var_1}{2} \\ var_3 &= \frac{var_1 + var_2}{2} = \frac{x + 3var_1}{4} \\ var_4 &= \frac{var_2 + var_3}{2} = \frac{3x + 5var_1}{8} \\ var_5 &= \frac{var_3 + var_4}{2} = \frac{5x + 11var_1}{16} \end{aligned}$$

A table is constructed as before to easily picture the multipliers and denominator.

Table 2: Multipliers and Divisors variables given the first 2 values of the sequence are known

variable	x	var_1	denominator
var_2	1	1	2
var_3	1	3	4
var_4	3	5	8
var_5	5	11	16

Immediately it is noticeable that the multipliers for x and var_1 are again the Jacobsthal Numbers and the denominator is 2 to the power of the position, n, minus 1.

$$var_n = \frac{J_{n-1}x + J_n var_1}{2^{n-1}} \tag{5}$$

3 GENERALISING INTO 1 EQUATION

The equations in Section §2 are already less compute intensive than having to solve the problem using Algorithm 1. However, to find a variable that is not the first 2 numbers of the sequence, 2 computations are required. What if the equations can be simplified into 1 equation that solves for all unknown variables?

At the beginning of the problem only the 2 boundary values are known. Thus the unknown variables need to be determined using the only 2 numbers provided. This leads to the following forumalæ for the variables ranging from 1 to 5.

$$\begin{array}{lll} |vars| = 1, & var_1 = 2y - x & \\ |vars| = 2, & var_1 = \frac{4y - x}{3} & var_2 = \frac{4y + 2x}{6} \\ |vars| = 3, & var_1 = \frac{8y - 3x}{5} & var_2 = \frac{8y + 2x}{10} \\ & var_3 = \frac{24y - 4x}{20} & \\ |vars| = 4, & var_1 = \frac{16y - 5x}{11} & var_2 = \frac{16y + 6x}{22} \\ & var_3 = \frac{48y - 4x}{44} & var_4 = \frac{80y + 8x}{88} \\ |vars| = 5, & var_1 = \frac{32y - 11x}{21} & var_2 = \frac{32y + 10x}{42} \\ & var_3 = \frac{96y - 12x}{84} & var_4 = \frac{160y + 43x}{168} \\ & var_5 = \frac{352y - 27x}{336} & \end{array}$$

Constructing a table as done in the previous section will be of little help here. Nonetheless some patterns can be immediately spotted. First the denominator for variables var_2, var_3, var_4 and var_5 is a multiplier of the denominator of var_1. Secondly, the sign in the numerator formula corresponds to -1^n where n is the position of the variable in the sequence. It is also noticeable that the multiplier of y follows the Jacobsthal Numbers for n multiplied by $2^{|vars|}$.

The multiplier for x, is slightly more complex. Besides the changing sign it looks to be going up and down with each variable. Plotting the multipliers of x in a table:

Table 3: Multipliers of x

variable	n = 1	n = 2	n = 3	n = 4	n= 5
a	-1	-1	-3	-5	-11
b		2	2	6	10
c			-4	-4	-12
d				8	43
e					-27

A pattern can be seen besides the change in the sign. The value is going up and down following a rhythm. With a little more inspection, the changes can be determined to follow a computation on the Jacobsthal Number sequence.

Based on this analysis, the generalised equation can be determined to be:

$$var_n = \frac{2^{|vars|}J_n y + \left(-1^n J_{|vars|} + -1^{|vars|} J_{n-1}\right) x}{J_{|vars|+1} 2^{n-1}} \tag{6}$$

4 PROOFS

In the previous 2 sections; 3 equations where presented as a solution to the original imposed questions. Intuitively the equations look to hold for the sample of data taken into consideration.

In Section §2 the first unknown variable was determined to satisfy Equation 4. On the other hand, in Section §3 a different formula was provided to find the same value.

THEOREM 4.1. *Equation 4 is identical to Equation 6 for n = 1*

PROOF.

- Given
$$var_n = \frac{2^{|vars|}J_n y + \left(-1^n J_{|vars|} + -1^{|vars|}J_{n-1}\right)x}{J_{|vars|+1}2^{n-1}}$$
- Distributive Property of Equality
$$var_n = \frac{2^{|vars|}J_n y + -1^n J_{|vars|}x + -1^{|vars|}J_{n-1}x}{J_{|vars|+1}2^{n-1}}$$
- Let n = 1
$$var_1 = \frac{2^{|vars|}J_1 y + -1^1 J_{|vars|}x + -1^{|vars|}J_{1-1}x}{J_{|vars|+1}2^{1-1}}$$
- Simplification for powers
$$var_1 = \frac{2^{|vars|}J_1 y - J_{|vars|}x + -1^{|vars|}J_0 x}{J_{|vars|+1}}$$
- $J_1 = 1$ and $J_0 = 0$
$$var_1 = \frac{2^{|vars|}y - J_{|vars|}x}{J_{|vars|+1}}$$
- The end result is Equation 4

□

Before proceeding with determining if Equation 6 is a generalisation based on the Arithmetic Mean, Equation 1, it is important to remind about the following lemma.

LEMMA 4.2. *For any* $n \in \mathbb{Z}$, $-1^n \equiv 2(-1^{n-2} + -1^{n-1})$

PROOF.

- Given
$$2(-1^{n-2} + -1^{n-1})$$
- Changing for -1^{n-2} common
$$-1^{n-2}(2 + -1^{-1})$$
- Simplification
$$-1^{n-2}(2 + -1)$$
- Simplification with the rule $\forall n | n \in \mathbb{Z}, -1^{n-2} = -1^n$
$$-1^n$$

□

THEOREM 4.3. *Given the first 2 numbers of the sequence, x and* var_1*. Equation 5 is the a generalisation of the Arithmetic Mean of the previous 2 values*

PROOF.

- Given
$$var_n = \frac{J_{n-1}x + J_n var_1}{2^{n-1}}$$
- Show true for $n = 2$
$$\begin{aligned} var_2 &= \frac{J_1 x + J_2 var_1}{2^{n-1}} \\ &= \frac{x + var_1}{2} \end{aligned}$$
- Show true for $n = 3$
$$\begin{aligned} var_3 &= \frac{J_{n-1}x + J_n var_1}{2^{n-1}} \\ &= \frac{J_2 x + J_3 var_1}{2^2} \\ &= \frac{x + 3var_1}{4} \\ &= \frac{\left(\frac{x+var_1}{2}\right) + \left(\frac{2var_1}{2}\right)}{2} \\ &= \frac{var_2 + var_1}{2} \end{aligned}$$
- Suppose true for var_n
- Show true for var_{n+1}
$$\begin{aligned} var_{n+1} &= \frac{J_n x + J_{n+1} var_1}{2^n} \\ &= \frac{J_n x + (2J_{n-1} + J_n) var_1}{2^n} \\ &= \frac{J_n x + 2J_{n-1} var_1 + J_n var_1}{2^n} \\ &= \frac{2J_{n-2}x + J_{n-1}x + 2J_{n-1}var_1 + J_n var_1}{2^n} \\ &= \frac{\frac{2(J_{n-2}x + J_{n-1}var_1)}{2^{n-1}} + \frac{J_{n-1}x + J_n var_1}{2^{n-1}}}{2} \\ &= \frac{\frac{J_{n-2}x + J_{n-1}var_1}{2^{n-2}} + var_n}{2} \\ &= \frac{var_{n-1} + var_n}{2} \end{aligned}$$

□

THEOREM 4.4. *Equation 6 is the a generalisation of the Arithmetic Mean of the previous 2 values*

PROOF.

- Given
$$var_n = \frac{2^{|vars|}J_n y + \left(-1^n J_{|vars|} + -1^{|vars|}J_{n-1}\right)x}{J_{|vars|+1}2^{n-1}}$$
- Distributive Property of Equality
$$var_n = \frac{2^{|vars|}J_n y + -1^n J_{|vars|}x + -1^{|vars|}J_{n-1}x}{J_{|vars|+1}2^{n-1}}$$
- Using Lemma 4.2, $-1^n \equiv 2(-1^{n-2} + -1^{n-1})$
$$var_n = \frac{\begin{pmatrix} 2^{|vars|}J_n y + 2(-1^{n-2} + -1^{n-1})J_{|vars|}x + \\ -1^{|vars|}J_{n-1}x \end{pmatrix}}{J_{|vars|+1}2^{n-1}}$$
- Distributive Property of Equality
$$var_n = \frac{\begin{pmatrix} 2^{|vars|}J_n y + 2(-1^{n-2})J_{|vars|}x + \\ -1^{n-1}J_{|vars|}x + -1^{|vars|}J_{n-1}x \end{pmatrix}}{J_{|vars|+1}2^{n-1}}$$

- Expanding for J_n amd J_{n-1}

$$var_n = \frac{\begin{pmatrix} 2^{|vars|}(2J_{n-2} + J_{n-1})y + 2(-1^{n-2})J_{|vars|}x + \\ -1^{n-1}J_{|vars|}x + -1^{|vars|}(2J_{n-3} + J_{n-2})x \end{pmatrix}}{J_{|vars|+1}2^{n-1}}$$

- Distributive Property of Equality

$$var_n = \frac{\begin{pmatrix} 2\left(2^{|vars|}J_{n-2}y\right) + 2\left(-1^{n-2}J_{|vars|}x\right) + \\ 2\left(-1^{|vars|}J_{n-3}x\right) + 2^{|vars|}J_{n-1}y + \\ -1^{n-1}J_{|vars|}x + -1^{|vars|}J_{n-2}x \end{pmatrix}}{J_{|vars|+1}2^{n-1}}$$

- Expanding the equation and extracting the division by 2

$$var_n = \frac{\begin{pmatrix} \frac{2^{|vars|}J_{n-2}y + \left(-1^{n-2}J_{|vars|} + -1^{|vars|}J_{n-3}\right)x}{J_{|vars|+1}2^{n-3}} + \\ \frac{2^{|vars|}J_{n-1}y + \left(-1^{n-1}J_{|vars|} + -1^{|vars|}J_{n-2}\right)x}{J_{|vars|+1}2^{n-2}} \end{pmatrix}}{2}$$

- Using Equation 6 for simplification

$$var_n = \frac{var_{n-2} + var_{n-1}}{2}$$

□

By induction from Theorems 4.1 and 4.4, Lemma 4.5 can be concluded.

LEMMA 4.5. *Equation 4 holds for $n = 1$ given any number of unknown variables in the sequence, $|vars|$.*

5 CONCLUSION

Given an arithmetic mean sequence, where only the first and last numbers in the sequence are known, it is possible to find the other values in the sequence. Using simultaneous equations and recursive replacement of variables it is possible to solve the sequence. For decently sized lists, this is a time consuming and prone to errors. This paper sets out to solve the time consuming problem while also reduce the complexity of the solution. The solution was achieved by initially a small set of simultaneous equations and the recursive replacements for the range of unknown variables ranged between 1 to 5 was solved in terms of the unknown values. The obtained expressions where then analysed and 3 linear equations - Equations 4, 5 and 6 - where obtained. As the equations were obtained through observation of a small set, formal proofs were obtained for each equation to prove their general applicability for the problem.

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