Optimisation to Geometric Mean Sequence Intermediate Values Solver

Research

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ABSTRACT

Sequences are easily generated when the first 2 or more numbers in the sequence are known and a pattern, formula, is identified as the generating function. However it gets complicated when only the first 2 numbers of a sequence are known. For example, consider a scenario where a sequence of values is presented and one needs to find intermediate values between any two consecutive values in the sequence. The original sequence is known to follow the Geometric Mean progression, thus the intermediate value are assumed to follow the Geometric Mean progression as well. By taking a finite number of intermediate values between the selected numbers, once can use simultaneous equations to find the required intermediate values. However solving simultaneous equations is not linearly computational. This paper provides 3 different formulæ that optimise the computation to a linear algorithm.

KEYWORDS

Average Sequence, Geometric Mean Sequence, Optimisation Formulæ, Geometric Mean Special Case

1 INTRODUCTION

A Geometric Mean sequence can be generated by using Equation 1 as the generating function. For example, the geometric mean sequence for a list of numbers starting with 9 and 16 will look like

$$5, 100, 22.36068, 47.28708, 32.51725, 39.21282, 35.70845$$

$$\texttt{Geometric Mean} = \sqrt{ab} \tag{1}$$

Consider the scenario where the above sequence are points on an oscillating behaviour. More information is required to be provided around the first 2 observation points, 5 and 100. Assuming that the behaviour follows the Geometric Mean, determine the 10 values between the 5 and 100 points. One can proceed with generating the below sequence and solve the values using simultaneous equations. With 10 unknowns it is already a tedious exercise prone to errors. Now scale up the computations to 100 and the complexity will exponentially grow.

$$5, a, b, c, d, e, f, g, h, i, j, 100$$

This paper presents a set of computationally optimised forumulæ to solve the value for a finite set of intermediate values.

2 FINDING THE VALUE FOR v_1

In order to determine a formula for calculating the first unknown in the sequence, a number of simultaneous equations need to be calculated to determine if a pattern exists. Exhibit 1

$$|var| = 1, \qquad var_1 = \frac{b^2}{a}$$

$$|var| = 2, \qquad var_1 = \sqrt[3]{\frac{b^4}{a}}$$

$$|var| = 3, \qquad var_1 = \sqrt[5]{\frac{b^8}{a^3}}$$

$$|var| = 4, \qquad var_1 = \sqrt[11]{\frac{b^{16}}{a^5}}$$

$$|var| = 5, \qquad var_1 = \sqrt[21]{\frac{b^{32}}{a^{11}}}$$

Exhibit 1: Values for v_1 for varying sequence lengths

Analysing the data, it is observable that the value can be generalised to Equation 2

$$v_1 = \sqrt[J_{|var|+1}]{\frac{b^{2^{|var|}}}{a^{J_{|var|}}}} \tag{2}$$

Interestingly the square root factor increases following the Jacobsthal sequence, [1]. Similarly the value of a, the first known number in the sequence, also increases following the Jacobsthal sequence. But lays one value behind the square root factor.

3 FINDING THE n^{th} VALUE GIVEN THE FIRST 2 NUMBERS IN SEQUENCE

Given that the first unknown value is found, the remaining variables in the sequence can be determined. Exhibit 2 shows that a formula can be generated to solve the variables directly.

$$|var| = 2, \quad var_2 = \sqrt{a \cdot var_1}$$

$$|var| = 3, \quad var_2 = \sqrt{a \cdot var_1} \quad var_3 = \sqrt[4]{a \cdot var_1^3}$$

$$|var| = 4, \quad var_2 = \sqrt{a \cdot var_1} \quad var_3 = \sqrt[4]{a \cdot var_1^3} \quad var_4 = \sqrt[8]{a^3 \cdot var_1^5}$$

$$|var| = 5, \quad var_2 = \sqrt{a \cdot var_1} \quad var_3 = \sqrt[4]{a \cdot var_1^3} \quad var_4 = \sqrt[8]{a^3 \cdot var_1^5} \quad var_5 = \sqrt[16]{a^5 \cdot var_1^{11}}$$

Exhibit 2: Values for v_n for varying sequence lengths

Based on the data in Exhibit 2, formula 3 can be observed.

$$var_n = \sqrt[2^{n-1}]{a^{J_{n-1}} v_1^{J_n}} \tag{3}$$

4 FINDING THE OTHER VARIABLES IN TERMS OF a AND b

Having to find the first unknown variable value to solve a specific unknown is not always ideal. Equation 2 provides a means to replace var_1 in terms of the 2 known values in the sequence. Thus Exhibit 3 can be generated from Exhibit 2.

$$|var| = 2, \quad var_2 = \sqrt[3]{ab^2}$$

$$|var| = 3, \quad var_2 = \sqrt[5]{ab^4} \quad var_3 = \sqrt[5]{\frac{b^6}{a}}$$

$$|var| = 4, \quad var_2 = \sqrt[11]{a^3b^8} \quad var_3 = \sqrt[11]{\frac{b^{12}}{a}}$$

$$var_4 = \sqrt[11]{ab^{10}}$$

$$|var| = 5, \quad var_2 = \sqrt[21]{a^5b^{16}} \quad var_3 = \sqrt[21]{\frac{b^{24}}{a^3}}$$

$$var_4 = \sqrt[21]{ab^{20}} \quad var_5 = \sqrt[21]{\frac{b^{22}}{a}}$$

Exhibit 3: Repertoire for unknown variables in sequence based on known values

Generalising the values in Exhibit 3, formula 4 can be obtained.

$$v_n = \sqrt[J_{|var|+1}]{b^{\kappa(|var|,n)} a^{-1^n J_{(|vars|-n+1)}}} \tag{4}$$

5 DEFINING THE FUNCTION $\kappa(\lambda, n)$

In Equation 4 a new function κ is introduced to simplify the writing of equation. The function κ is defined as

$$\kappa(\lambda, n) \stackrel{\text{def}}{=} 2^\lambda \left(\frac{J_n}{2^{n-1}}\right) \tag{5}$$

The Jacobsthal function can be written in $O(1)$, using Binet formula [2, 3]. Thus the κ function can be computed in $O(1)$.

$$\kappa(\lambda, n) \stackrel{\text{def}}{=} 2^\lambda \left(\frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\right) \tag{6}$$

5.1 Properties of $\kappa(\lambda, n)$ function

A number of properties have been observed on the κ function.

5.1.1 Values for $\kappa(\lambda, n)$ for $1 \le n \le \lambda$.
Generating the values for $\kappa(\lambda, n)$ where $\lambda \in [1, 8]$ an interesting pattern is observed

λ	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
01 :	2.0							
02 :	4.0	2.0						
03 :	8.0	4.0	6.0					
04 :	16.0	8.0	12.0	10.0				
05 :	32.0	16.0	24.0	20.0	22.0			
06 :	64.0	32.0	48.0	40.0	44.0	42.0		
07 :	128.0	64.0	96.0	80.0	88.0	84.0	86.0	
08 :	256.0	128.0	192.0	160.0	176.0	168.0	172.0	170.0

THEOREM 5.1. *When $n = 1$ the result is always 2^λ*

PROOF.

- Given definition

$$\kappa(\lambda, n) \stackrel{\text{def}}{=} 2^\lambda \left(\frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\right)$$

- $n = 1$

$$\kappa(\lambda, 1) = 2^\lambda \left(\frac{2^1 - (-1)^1}{3 \cdot 2^{1-1}}\right)$$

- Computing for values

$$\kappa(\lambda, 1) = 2^\lambda \left(\frac{2 + 1}{3 \cdot 0}\right)$$

- Simplifications

$$\kappa(\lambda, 1) = 2^\lambda$$

□

THEOREM 5.2. *When $n = \lambda$ the result is always $2J_\lambda$*

PROOF.

- Given definition

$$\kappa(\lambda, n) \stackrel{\text{def}}{=} 2^\lambda \left(\frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\right)$$

- $n = \lambda$

$$\kappa(\lambda, \lambda) = 2^\lambda \left(\frac{2^\lambda - (-1)^\lambda}{3 \cdot 2^{\lambda-1}}\right)$$

- $\frac{2^\lambda}{2^{\lambda-1}} = 2$

$$\kappa(\lambda, \lambda) = 2 \left(\frac{2^\lambda - (-1)^\lambda}{3}\right)$$

- Substituting back to Jacobsthal formula

$$\kappa(\lambda, \lambda) = 2J_\lambda$$

□

CONJECTURE 1. *The sequence can be generated by alternating the addition and subtraction of $2^{\lambda-(n-1)}$ from the previous value For example: Let $\lambda = 7$*

$$n = 1, \quad \kappa(7, 1) = 2^7 = 128$$

$$n = 2, \quad \kappa(7, 2) = \kappa(7, 1) - 2^{7-1} = 64$$

$$n = 3, \quad \kappa(7, 3) = \kappa(7, 2) + 2^{7-2} = 96$$

$$n = 4, \quad \kappa(7, 4) = \kappa(7, 3) - 2^{7-3} = 80$$

$$n = 5, \quad \kappa(7, 5) = \kappa(7, 4) + 2^{7-4} = 88$$

$$n = 6, \quad \kappa(7, 6) = \kappa(7, 5) - 2^{7-5} = 84$$

$$n = 7, \quad \kappa(7, 7) = \kappa(7, 6) + 2^{7-6} = 86$$

CONJECTURE 2. *Given the sequence for any λ it is possible to compute the values of $\lambda + 1$ directly*
The tree above provides up to $\lambda = 8$ without using any formul,a the values of $\lambda = 9$ can be constructed

$$
\begin{aligned}
n = 1, \quad & Theorem\ 5.1 = 512\\
n = 2, \quad & \kappa(8,0) + \kappa(8,1) = 256\\
n = 3, \quad & \kappa(8,1) + \kappa(8,2) = 384\\
n = 4, \quad & \kappa(8,2) + \kappa(8,3) = 320\\
n = 5, \quad & \kappa(8,3) + \kappa(8,4) = 352\\
n = 6, \quad & \kappa(8,4) + \kappa(8,5) = 336\\
n = 7, \quad & \kappa(8,5) + \kappa(8,6) = 344\\
n = 8, \quad & \kappa(8,6) + \kappa(8,7) = 340\\
n = 9, \quad & \kappa(8,7) + \kappa(8,8) = 342
\end{aligned}
$$

5.1.2 $\kappa(\lambda, 0)$ is always equal to 0. The value of the function $\kappa(\lambda, 0)$ is always 0 since the Jacobsthal Sequence for $n = 0 = J_0 = 0$.

THEOREM 5.3. *When $n = 0$ the result is always* 0

PROOF.

- Given definition
$$\kappa(\lambda, n) \stackrel{\text{def}}{=} 2^\lambda \left(\frac{J_n}{2^{n-1}}\right)$$
- $n = 0, J_0 = 0$
$$\kappa(\lambda, 0) = 2^\lambda \left(\frac{0}{3 \cdot 2^{0-1}}\right)$$
- Anything multiplied by 0 is 0
$$\kappa(\lambda, 0) = 0$$

□

5.1.3 Limits of function in infinite sequence.

PROPOSITION 5.4. *The series $\kappa(\lambda, n) \stackrel{def}{=} 2^\lambda \left(\frac{J_n}{2^{n-1}}\right)$ converges to $\frac{2^{\lambda+1}}{3}$*

PROOF.

- Given definition
$$\kappa(\lambda, n) \stackrel{\text{def}}{=} 2^\lambda \left(\frac{J_n}{2^{n-1}}\right)$$
- Finding the limit of sequence when n approaches ∞
$$\lim_{n\to\infty} \kappa(\lambda, n) = \lim_{n\to\infty} 2^\lambda \left(\frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\right)$$
- Applying the product of limits
$$\lim_{n\to\infty} \kappa(\lambda, n) = 2^\lambda \lim_{n\to\infty} \left(\frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\right)$$
- Extracting 2^n as common for numerator
$$\lim_{n\to\infty} \kappa(\lambda, n) = 2^\lambda \lim_{n\to\infty} \left(\frac{2^n(1 - \frac{(-1)^n}{2^n})}{3 \cdot 2^{n-1}}\right)$$
- Simplifying for $\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$
$$\lim_{n\to\infty} \kappa(\lambda, n) = 2^\lambda \lim_{n\to\infty} \left(\frac{2^n}{3 \cdot 2^{n-1}}\right)$$
- Simplification
$$\lim_{n\to\infty} \kappa(\lambda, n) = \frac{2^{\lambda+1}}{3}$$

□

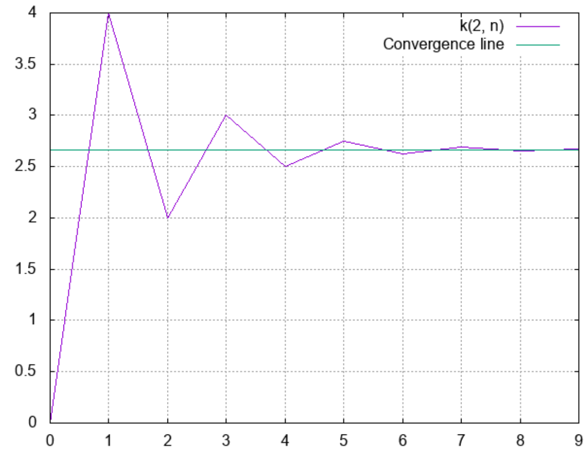

Figure 1: Convergence of $\kappa(2, n)$

PROPOSITION 5.5. *The series $\kappa(\lambda, n) \stackrel{def}{=} 2^\lambda \left(\frac{J_n}{2^{n-1}}\right)$ diverges to negative infinity $(-\infty)$ as $n \longrightarrow -\infty$*

For example: Let $\lambda = 2$

$$
\begin{aligned}
n = -1, \quad & \kappa(2,-1) = \kappa(2,0) + 2^{2+|-1|} &= 8\\
n = -2, \quad & \kappa(2,-2) = \kappa(2,-1) - 2^{2+|-2|} &= -8\\
n = -3, \quad & \kappa(2,-3) = \kappa(2,-2) + 2^{2+|-3|} &= 24\\
n = -4, \quad & \kappa(2,-4) = \kappa(2,-3) - 2^{2+|-4|} &= -40\\
n = -5, \quad & \kappa(2,-3) = \kappa(2,-4) + 2^{2+|-5|} &= 88\\
n = -6, \quad & \kappa(2,-4) = \kappa(2,-5) - 2^{2+|-6|} &= -168
\end{aligned}
$$

PROOF.

- Given definition
$$\kappa(\lambda, n) \stackrel{\text{def}}{=} 2^\lambda \left(\frac{J_n}{2^{n-1}}\right)$$
- Finding the limit of sequence when n approaches ∞
$$\lim_{n\to-\infty} \kappa(\lambda, n) = \lim_{n\to-\infty} 2^\lambda \left(\frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\right)$$
- Applying the product of limits
$$\lim_{n\to-\infty} \kappa(\lambda, n) = 2^\lambda \lim_{n\to-\infty} \left(\frac{2^n - (-1)^n}{3 \cdot 2^{n-1}}\right)$$
- Extracting 2^n as common for numerator
$$\lim_{n\to-\infty} \kappa(\lambda, n) = 2^\lambda \lim_{n\to-\infty} \left(\frac{2^n(1 - \frac{(-1)^n}{2^n})}{3 \cdot 2^{n-1}}\right)$$
- Simplification of powers
$$\lim_{n\to-\infty} \kappa(\lambda, n) = 2^\lambda \lim_{n\to-\infty} \left(\frac{1-\infty}{2^\infty} \times \frac{2^{\infty-1}}{3}\right)$$
- Simplification
$$\lim_{n\to-\infty} \kappa(\lambda, n) = -\infty$$

□

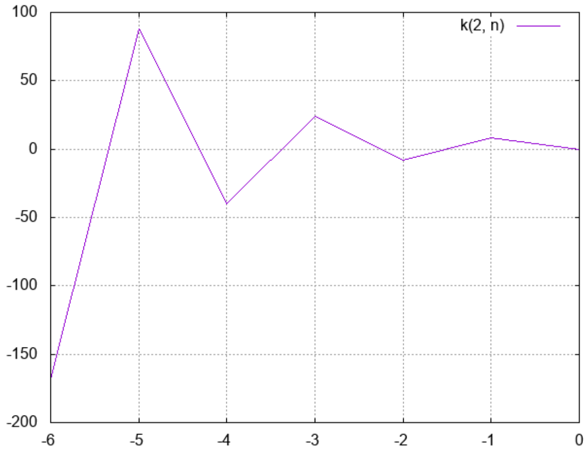

Figure 2: Divergence of $\kappa(2, n)$ for $n < 0$

6 VALIDATION OF GEOMETRIC MEAN SOLUTIONS

THEOREM 6.1. *Equation 4 is reducible to Equation 1 for $n = 1$*

PROOF.

- Given definition
$$v_n = \sqrt[J_{|var|+1}]{b^{\kappa(|var|,n)} a^{-1^n J_{(|vars|-n+1)}}}$$
- Let $n = 1$
$$v_1 = \sqrt[J_{|var|+1}]{b^{\kappa(|var|,1)} a^{-1^1 J_{(|vars|-n+1)}}}$$
- Theorem 5.1 and simplifications
$$v_1 = \sqrt[J_{|var|+1}]{\frac{b^{2^{|var|}}}{a^{J_{|vars|}}}}$$

□

CONJECTURE 6.2. *Equation 4 is reducible to Equation 3 for $n > 1$*

REFERENCES

[1] Wikipedia contributors. 2018. Jacobsthal number — Wikipedia, The Free Encyclopedia. Retrieved March 19, 2019 from https://en.wikipedia.org/w/index.php?title=Jacobsthal_number&oldid=849904767

[2] Julius Fergy T. Rabago. 2013. A Note on Modified Jacobsthal and Jacobsthal–Lucas Numbers. *Notes on Number Theory and Discrete Mathematics* 19, 3 (2013), 15–20. Retrieved March 19, 2019 from http://nntdm.net/papers/nntdm-19/NNTDM-19-3-15-20.pdf

[3] Anetta Szynal-Liana and Iwona Włoch. 2016. A Note on Jacobsthal Quaternions. *Advances in Applied Clifford Algebra*, Article 26 (2016), 441–447 pages. Retrieved March 26, 2019 from https://core.ac.uk/download/pdf/81813252.pdf